MATHEMATICS OF FINANCE – PART 2

ASHA RANI

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BUSINESS MATHEMATICS (PAPER- BCH 4.2), SECTION A

Session Details

This session covers the following:

- Meaning of annuity
- Types of annuity
- Ordinary annuity- future and present value
- Application of ordinary annuity for various purpose

Meaning of Annuity

Annuity is a series of equal payments made at usually equal interval of time or in other word an annuity is a periodic amount of money that is paid at regular intervals

The concept of annuity can be used to turn the one time lump sum payment into payments of smaller denomination in future time period for our convenience.

Examples of annuity:

- **Instalment payment – leasing agreement, Repayment of loan**
- **Regular deposits to saving bank account**
- **Insurance premium**
- **Hire purchase agreements etc**.

Types of annuity

- Annuity
  - Ordinary Annity
  - Annuity Due
  - Deferred annuity
  - Perpetuity

Ordinary annuity

Ordinary annuity is defined as the annuity where the payment is made at the end of each payment interval.

Ordinary annuity is series of payments which possess the following characteristics:

1) Amount of payment should be same
2) Interval time between the payment should be same
3) Payment should be made at the end of each period

Ordinary annuity

Future value of ordinary annuity – refers to the value that is compounding till the end of its term/duration or it is also defined as the sum of future value of all the periodic payments at the end of annuity.

Present value of ordinary annuity – refers to the sum of discounted value of each periodic payment at the given rate of interest. Also termed as capital value.

Future value of ordinary annuity

The formula for the **future value** of an **ordinary annuity**

$$FV_n = R\ [\ ((1+i)^n - 1)]/i$$

- OR

$$FV_n = R\ s_n \neg i$$

- Where,
- FV_n =Future value of annuity at the end of each period
- R= Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts
- $s_n \neg i$ = tabulated value of future value of Interest rate factor annuity (FVIFA)

Derivation of Future value of ordinary annuity will be given in the next slide

Suppose R payments are made for n number of years at $r\%$ compounded annually, then on the time line the series of payments can be depicted as under:

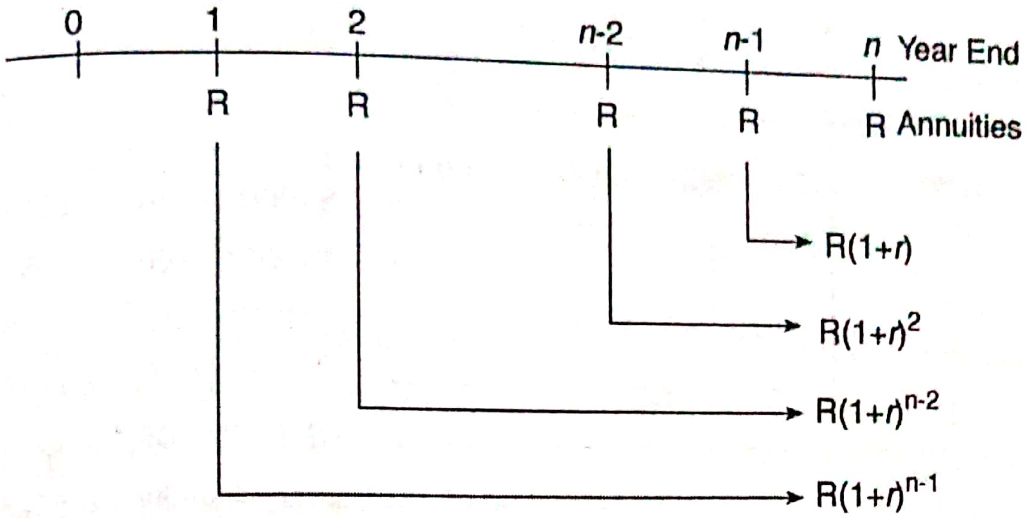

Figure 12.1

The 1st payment made at the end of 1st year compounds to $R(1+r)^{n-1}$ for $(n-1)$ years ; The 2nd payment made at the end of 2nd year compounds to $R(1+r)^{n-2}$ for $(n-2)$ years ; The $(n-2)^{th}$ payment made at the end of $(n-2)^{th}$ year compounds to $R(1+r)^2$; $(n-1)^{th}$ payment made at the end of $(n-1)^{th}$ year compounds to $R(1+r)$ and the last payment made at the end of n^{th} year does not compound and its value at the end of period remains unchanged. The sum of all the compounded payments (denoted by S) is expressed as below:

$$S = R + R(1+r) + R(1+r)^2 + \ldots\ldots\ldots\ldots + R(1+r)^{n-2} + R(1+r)^{n-1}$$

This is a geometric series, which results in the following summation:

$$S = R\left[\frac{(1+r)^n - 1}{(1+r) - 1}\right] = R\left[\frac{(1+r)^n - 1}{r}\right]$$

The above formula computes the amount (or future value) of annuity R, being the series of payments that have been compounded at the r% per period for n number of periods.

Further the expression:

$\left[\frac{(1+r)^n - 1}{r}\right]$ is denoted by $S_{\overline{n}|r\%}$ and is called the Future Value of Interest Factor Annuity (FVIFA), for which a double precision table has been compiled and given as appendix 6. Accordingly,

$$S = R \times S_{\overline{n}|r\%}$$

Using above formula, the amount of annuity can be obtained, given the annuity and the similarly annuity can be obtained for a given future value corresponding to such annuity for a given rate of interest and number of periods.

Present value of ordinary annuity

The formula for the Present **value** of an **ordinary annuity**

$$\mathrm{PV}_n = R\,[1 - ((1+i)^{-n})]/i$$

- OR

$$\mathrm{PV}_n = R\, a_n \neg i$$

- Where,
- FV_n =Future value of annuity at the end of each period
- R= Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts
- $a_n \neg i$ = tabulated value of present value of Interest rate factor annuity (PVIFA)

Derivation of Present value of ordinary annuity given in the next slide

Present value (or current value) of annuity, also called capital value of an annuity refers to the sum of the periodic payments each discounted at the given rate of interest to reflect the time value of money. The present value of an annuity is the amount which if invested at the start of first period at the given rate of interest will be sufficient to pay off each of the annuities out of such investment as and when they become due.

The aggregate of annuities duly discounted till the end of its duration gives the present value of such annuities. Suppose R payments are made for *n* number of years at *r*% discounted annually, then on the time line the series of payments can be depicted as under:

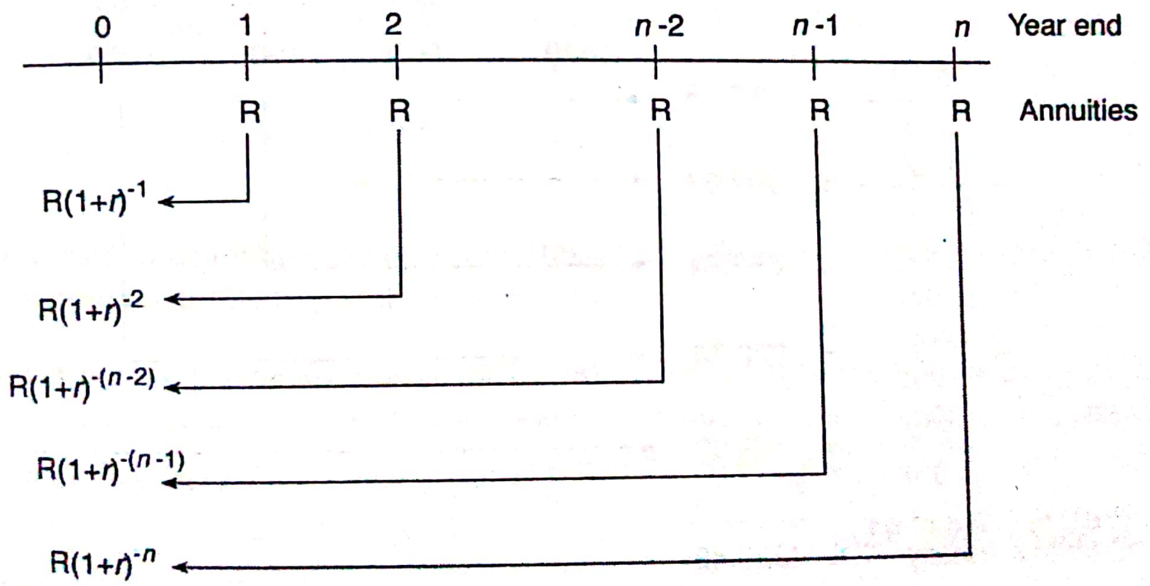

Figure 12.3

The 1st payment made at the end of 1st year discounts to $R\ (1+r)^{-1}$; The 2nd payment made at the end of 2nd year discounts to $R(1+r)^{-2}$; The $(n-2)$th payment made at the end of $(n-2)^{th}$ year discounts to $R(1+r)^{-(n-2)}$; $(n-1)$th payment made at the end of $(n-1)^{th}$ year discounts to $R(1+r)^{-(n-1)}$ and the last payment made at the end of nth year is not discounted and its value at the start of period discounts to $R\ (1+r)^{-n}$. The sum of all the discounted payments (denoted by P) is expressed as below:

$$P = R(1+r)^{-1} + R(1+r)^{-2} + \dots\dots\dots\dots + R(1+r)^{-(n-2)} + R(1+r)^{-(n-1)} + R(1+r)^{-n}$$

This is a geometric series, which results in the following summation:

$$P = R(1+r)^{-1}\left[\frac{1-(1+r)^{-n}}{1-(1+r)}\right] = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

The above formula computes the present value of annuity R, being the series of payments that have been compounded at the $r\%$ per period for n number of periods.

Further the expression:

$\left[\dfrac{1-(1+r)^{-n}}{r}\right]$ is denoted by $A_{\overline{n}|r\%}$ and is called the Present Value of Interest Factor Annuity (PVIFA), for which a double precision table has been compiled and given as appendix 4. Accordingly,

$$P = R \times A_{\overline{n}|r\%}$$

Using above formula, the present value of annuity can be obtained given the annuity and similarly

ILLUSTRATIVE PROBLEMS

1) At six-month interval, A deposited ₹2000 in a saving account which credit interest at 10% p.a. compounded semi-annually. The first deposit was made when A' s son was six-month-old and the last deposit was made when his son was 8 years old. The money remained in the account and was presented to the son on his 10^{th} birthday. How much did he receive?

2) Mr. X purchases a house for ₹2, 00,000. He agrees to pay for the house in 5 equal installments at the end of each year. If the money is worth 5% p.a. effective, what would be size of each investment? In case X makes a down payment of ₹50, 000 what would be the size of each installment?

Sol^n:- 1.

Firstly we have to find the amount accumulated in A's son account when he was 8 years old. for that we will apply F.V of ordinary annuity formula as A deposited ₹ 100 every six months for 8 years.

$$F.V = R\left[\frac{(1+i)^n - 1}{i}\right] \text{ OR } R\, s_{\overline{n}|i}$$

here $n = 16$

$i = 10\%$ semi-annually i.e 5%

$$= 100\, s_{\overline{16}|.05}$$

$$F.V = 100 \times (23.6574) = ₹\ 2365.74$$

Since this amount remained in the Account so he will get interest @ 10% p.m compounded semi-annually on this amount. ∴ this amount after two year will be

$$= 2365.74\left(1 + \frac{.10}{2}\right)^4 \Rightarrow 2365.74 \times 1.215506$$

$$\Rightarrow ₹\ 2875.57$$

Sol^n:-2 Let the size of monthly payments be ₹ R.

$$\underset{(P.P)}{200000} = R\, a_{\overline{n}|i} \Rightarrow R = \frac{200000}{a_{\overline{5}|.05}}$$

$$= \frac{200000}{4.32947667}$$

If X is making down payment of Rs. 50000

$$\boxed{R = ₹\ 46195.78}$$

Then

$$200000 - 50000 = R\, a_{\overline{n}|i}$$

$$R = \frac{150000}{a_{\overline{5}|.05}}$$

$$\boxed{R = ₹\ 34,646.83}$$

Amortization of loan

Amortization refers to the repayment of loan through a fixed repayment schedule in regular payment over a period of time. Each payment includes both interest on the outstanding amount of loan and principal amount.

This is done by applying the concept of *present value of an annuity*. Suppose a loan of Rs. A has been taken at a interest rate i % which is to be repaid in n regular payment , payable at the end of each payment interval , then the value of R regular installment can be obtained as follows:

- A= $R\ [1 - ((1+i)^{-n})]/i$
- A= $R\ a_n \neg i$
- R = A $/\ a_n \neg i$

  A loan is amortized if both the capital and interest are paid by a sequence of periodic payments.

Amortization Formulas

1. Periodic payment: $R = \dfrac{A}{a_{\overline{n}\rfloor r}} = A \cdot \dfrac{r}{1-(1+r)^{-n}}$
2. Principal outstanding at beginning of kth period: $Ra_{\overline{n-k+1}\rfloor r} = R \cdot \dfrac{1-(1+r)^{-n+k-1}}{r}$
3. Interest in kth payment: $Rr\,a_{\overline{n-k+1}\rfloor r}$
4. Principal contained in kth payment: $R(1 - r\,a_{\overline{n-k+1}\rfloor r})$
5. Total interest paid: $R(n - a_{\overline{n}\rfloor r}) = nR - A$

ILLUSTRATIVE PROBLEM

3. Mr. X took a loan of ₹80,000 payable in 10 semiannual installments, rate of interest being 8% p.a. compounded semiannually, find:

- The amount of each installment;
- Loan outstanding after 4^{th} payment;
- Interest component of 5^{th} payment; and
- Loan repaid after four payments.

Solⁿ:- 3.

(I) The amount of each instalment

$R = \frac{A}{a_{\overline{n}|i}}$ → amount of Loan; i → rate of interest; n → no. of semi-annual payments

$R = \frac{80,000}{a_{\overline{10}|.04}} = ₹\,9863.27$

(II) Loan o/s after 4th payment

$= R\,a_{\overline{n-k+1}|i} = R\,a_{\overline{10-5+1}|.04}$

$= 9863.27 \times a_{\overline{6}|.04}$ → take this value from table

$= 9863.27 \times 5.24213686$

$= ₹\,51,706.61$

(III) Interest component of 5th payment

$= R\,a_{\overline{n-k+1}|i} \times i$

$= 51,706.61 \times .04$

$= ₹\,2068.18$

(IV) Loan Repaid after Four Payments

$=$ Loan Amount $-$ Principal o/s at the beg. of 5th payment

$= 80,000 - R\,a_{\overline{10-5+1}|.04}$

$= 80,000 - 51,706.61$

$= ₹\,28,295.39$

LEASING DECISIONS

Once a firm has evaluated the economic viability of an asset as an investment and accepted/selected the proposal, it has to consider alternate methods of financing the investment

the firm may consider leasing of the asset rather than buying it. Hence, lease financing decisions relating to leasing or buying options primarily involve comparison between the cost of debt-financing and lease financing.

Evaluation of lease financing decisions involves the following steps:

- (i) Calculate the present value of net-cash flow of the buying option, called NPV (B).
- (ii) Calculate the present value of net cash flow of the leasing option, called NPV (L)
- (iii) Decide whether to buy or lease the asset or reject the proposal altogether by applying the following criterion:
- (a) If NPV (B) is positive and greater than the NPV (L), purchase the asset.

Capital expenditure decisions

In capital expenditure decisions a company has to make choice between two machines, both can be used to improve operation by saving in labour costs.

Given the time value of money , we can use the concept of annuity to determined the net annual savings of each machine and then decide which machine to buy.

ILLUSTRATIVE PROBLEM

4) Machine A costs ₹ 10,000 and has a useful life of 8 years. Machine B costs ₹8000 and has a useful life of 6 years. Suppose machine a generates an annual labour saving of ₹ 2000 which machine B generate an annual saving of ₹1800. Assuming the time value of money is 10% p.a., find which machine is preferable?

Solⁿ:-4 Machine A costs Rs. 10000, useful life - 8 yrs
Machine B costs Rs. 8000, useful life - 6 yrs
Suppose A generate annual labour saving of ₹ 2000, while B generates annual saving of ₹ 1800. Find which machine is preferable. Eqt - equivalent

Eqt Annual cost of Machine A By applying P.V concept

$$10000 = C_A \, a_{\overline{8}|.10}$$

$$C_A = \frac{10000}{5.335} = 1874.41$$

Eqt Annual cost of Machine B

$$8000 = C_B \, a_{\overline{6}|.10}$$

$$C_B = \frac{8000}{4.356} = 1836.55$$

We will compare
→ Annual savings for two Machine & Annual cost of two machines

	A	B
cost	1874.41	1836.55
Savings	2000.00	1800.00
	+ 125.59	- ₹ 36.55

Savings will be +ve in case of Machine A ∴ Machine A is preferable.

Further, we can also do this question by comparing the p.v of a sequence of annual saving in labour.

Valuation of Bond

A bond is generally a security for a debt, in which the person who is issuing holds a debt against the person who has taken the loan and thereby is obliged to pay the interest and the principal amount.

Usually, bonds are issued for longer periods which are usually greater than one year and which upon maturity, will be paid upon the principal amount (redemption value) or the periodical interest.

The process of determining these bonds is called bond valuation. It is used to determine the theoretical price or fair price or intrinsic price of the bonds.

THE FORMULAE FOR COMPUTING THE VALUE OF BOND

$$V = D\, a_{n}\neg i + RV\,(1+i)^{-n}$$

- Where,
- D =the periodic dividend payment
- i = the yield rate per period
- RV= the redemption price
- v= the purchase price
- n = number of period before redemption

Illustrative Problem

5) A ₹1000 bond paying annual dividends at the 8.5% will be redeemed at par at the end of 10 years. Find the purchase price of this bond if the investor wishes a yield rate of 8%.

Sol^n:- 5.

Face value of Bond is ₹ 1000 and is redeemed at par

∴ R.V is also ₹ 1000, yield rate = .08

D = annual Dividend $= 1000 \times \frac{8.5}{100}$

$= ₹ 85$

no. of period before Redemption

$= 10$

P.V of all future dividends

$= R a_{\overline{n}|i}$

P.V. of the Redemption price

$= R.V (1+i)^{-n}$

Purchase Price $= R a_{\overline{n}|i} + R.V (1+i)^{-n}$

$= 85 a_{\overline{10}|.08} + 1000 (1+.08)^{-10}$

$= 85 \times 6.7100 + 1000 \times (1.08)^{-10}$

$= 570.35 + 463.19$

$\boxed{\text{P.P. of Bond} = ₹ 1033.54}$

CONTINUOUS COMPOUNDED ANNUITY

Continuous compounding is compounding that is constant. One way some try to visualize the concept of continuous compounding is that is fluid, constantly compounding moment by moment, as opposed to daily, monthly, quarterly, or annually.

The future value of annuity with continuous compounding formula is the sum of future cash flows with interest. The sum of cash flows with continuous compounding can be shown as

- $FV = R + R\,e^{r} + R\,e^{2r} + R\,e^{3r} + \cdots + R\,e^{r(n-1)}$
- This is considered a geometric series as the cash flows are all equal. The common ratio for this example is e^{r}. To solve this continuous compounding series summation will be denoted by integration. So the formula will be
- $FV_n = \int_0^n R\,e^{rt}\,dt$
- Similarly, $PV_n = \int_0^n Re^{-rt}\,dt$

ILLUSTRATIVE PROBLEMS

6) An annuity of ₹500 p.a. is flowing continuously for 10 years. Find its future value if the rate of interest is 10% compounded continuously

7) Find the capital value of a uniform income stream of ₹ R per year for m years, reckoning interest continuously at 100r% per year. What will be the result if income is forever?

Soln:- 6.

F.V of an annuity when interest rate is compounding continuously @ 10%.

$$F.V = \int_0^n Re^{rt}\,dt$$

$$= \int_0^{10} 500e^{.10\times t}\,dt$$

$$= 500\left.\frac{e^{.10}}{.10}\right]_0^{10}$$

$$= \frac{500}{.10}\left[e^{.10\times 10} - e^{.10\times 0}\right]$$

$$= 5000\,[2.7183 - 1]$$

$$= 5000 \times 1.7183 = ₹\ 8591.5$$

Soln:-7 Capital value / P.V $= \int_0^n Re^{-rt}\,dt$

here $n = m$ years

$r = 100r\,\%$

$$P = \int_0^m Re^{-rt}\,dt = R\left.\frac{e^{-rt}}{-r}\right]_0^m$$

$$= \frac{R}{r}\left(1 - e^{-rm}\right)$$

if income is forever

then $P = \lim_{m\to\infty} \frac{R}{r}\left(1 - e^{-rm}\right)$

$(\because e^{-\infty} = 0)$

$$P = \frac{R}{r}\left(1 - e^{-\infty}\right) = \frac{R}{r}$$

Questions for Ordinary Annuity

Ques:1. At six-month interval, A deposited ₹2000 in a saving account which credit interest at 10% p.a. compounded semi-annually. The first deposit was made when A' s son was six-month-old and the last deposit was made when his son was 8 years old. The money remained in the account and was presented to the son on his 10^{th} birthday. How much did he receive?

Ques:2. An annuity of ₹500 p.a. is flowing continuously for 10 years. Find its future value if the rate of interest is 10% compounded continuously.

Ques:3. Mr. X deposits in his son's account ₹1000 times his son age at the end of each birthday. Find the balance accumulated at the 10^{th} birthday, if the rate of interest is 10% p.a. compounded annually.

Ques:4. A man requires ₹ 2, 00,000 to purchase a house after 5 years. He has an opportunity to invest the fund in an account which can earn 6% p.a. compounded quarterly. Find how much be deposited at the end of each quarter so as to have the required amount at the end of 5 years.

Ques:5. Mr. X purchases a house for ₹2, 00,000. He agrees to pay for the house in 5 equal installments at the end of each year. If the money is worth 5% p.a. effective, what would be size of each investment? In case X makes a down payment of ₹50, 000 what would be the size of each installment?

Ques:6. What should be the monthly sales volume of a company if it desires to earn 12% annual returns convertible monthly on its investment of ₹ 2,00,000? Monthly costs are ₹3, 000. The investment will have eight-year life with no scrap value?

Ques:7. Mr. X sells his old car for ₹ 100,000 to buy a new one costing ₹ 2,58,000. He pays ₹ x cash and balance by payment of ₹7000 at the end of each mount for 18 months. If the rate of interest is 9% compounded monthly, find x.

Ques:8. Find the capital value of a uniform income stream of ₹ R per year for m years, reckoning interest continuously at 100r% per year. What will be the result if income is forever?

Ques:9. According to an investment proposal, an initial investment of ₹ 1,00,000 is expected to yield a uniform income stream of ₹ 10,000 p.a. if the money is worth 8% p.a. compounded continuously, what is the expected payback period, i.e. after what time, the initial investment will be recovered?

Ques:10. If the present value and amount of an ordinary annuity of ₹1 p.a. for n years are ₹8.1109 and ₹12.0061 respectively, Find the rate of interest and the value of n without consulting the compound interest table.

Ques:11. Mr. X took a loan of ₹80,000 payable in 10 semiannual installments, rate of interest being 8% p.a. compounded semiannually, find:

1) The amount of each installment;
2) Loan outstanding after 4th payment;
3) Interest component of 5th payment; and
4) Loan repaid after four payments.

Ques:12. Mr. M borrowed ₹10, 00,000 from a bank to purchase a house and decided to repay by monthly equal installment in 10 years. The bank charges interest at 9% compounded monthly. The bank calculated his EMI as ₹12, 668. Find the principal and the interest paid in Ist and IInd year.

Ques:13. Machine A costs ₹ 10,000 and has a useful life of 8 years. Machine B costs ₹8000 and has a useful life of 6 years. Suppose machine a generates an annual labour saving of ₹ 2000 which machine B generate an annual saving of ₹1800. Assuming the time value of money is 10% p.a., find which machine is preferable?

Ques:14. Find the purchase of a ₹ 1000 bond, redeemable at the end of 10 years at ₹1100 and paying annual dividends at 4% if the yield rate is to be 5% p.a. effective.

Ques:15. A ₹1000 bond paying annual dividends at the 8.5% will be redeemed at par at the end of 10 years. Find the purchase price of this bond if the investor wishes a yield rate of 8%.